Special Issue

Mathematical Fluid Dynamics

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Introduction to Special Issue: Mathematical Fluid Dynamics

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Analysis of fluid flows has played a central role in the development of mathematical physics and applied mathematics since Newton's (1642–1727) original formulation of the linear constitutive relation for stress, which, when combined with his second law, led to the governing equations of Navier and Stokes. Euler's (1707–1783) subsequent introduction of the idea of a "material particle" set the stage for the "Lagrangian" formulation, which was further refined and exploited in Helmholtz's seminal paper (1858) on vortex motion,[1] while Prandtl's 1904 paper on the boundary layer theory led to the development of asymptotic methods widely used in other areas of mechanics where singular limits of dimensionless parameters form the starting point for general theories. Fluid mechanics aims to describe phenomena on a broad range of scales from biological flows at submicrometer lengths, to geophysical processes on kilometer scales, to astrophysical structures orders of magnitude beyond. For fully developed turbulent flows, excitations sustained by nonlinear processes over broad ranges of scales are active simultaneously. Its central role in the development of mathematical theory is perhaps best evidenced by the Clay Mathematics Institute's listing of "the existence and smoothness of solutions to the Navier-Stokes equations" as one of its six Millennium Prize Problems. It thus seems appropriate that a special issue of *Journal of Mathematical Physics* be devoted primarily to *mathematical* developments in the field.

The 24 original contributions to our special volume, written by over 50 leading experts and practitioners in the field, focus primarily on *incompressible* flows. We have divided the papers into five broad categories: (i) regularity results and analytical estimates for Navier-Stokes and related systems, (ii), vanishing viscosity limits and the relation between Navier-Stokes and Euler systems, (iii) mathematical issues related to discrete vortex and dynamical systems representations of fluid flows, (iv) statistical fluid dynamics and turbulence models, and (v) geophysical fluid dynamics models.

In the first category, **Regularity Results and Analytical Estimates for Navier-Stokes and Related Systems**, the manuscript of *R. Dascaliuc, C. Foias, and M. S. Jolly* considers limits, independent of viscosity, domain size, and strength of forcing, on solutions of the Navier-Stokes equations projected onto a normalized, dimensionless energy-enstrophy plane, obtaining an optimized bound on the parabola which must lie above the attractor. *J. D. Gibbon and G. A. Pavliotis* consider the two-dimensional Navier-Stokes equations on a periodic domain and obtain estimates for quantities such as long-time averages in terms of the Reynolds number, extending previous works which focused on these estimates in terms of the Grashof number. They also obtain estimates which shed light on the issue of intermittency. *I. Kukavica and M. Ziane* consider sufficient

[1]Euler's contributions are being celebrated in a 2007 workshop entitled "Euler Equations: 250 Years On" (URL: http://www.obs-nice.fr/etc7/EE250/), while Helmholtz's are being celebrated in 2008 in an IUTAM Symposium entitled "150 Years of Vortex Dynamics" (URL: http://www.fluid.dtu.dk/).

conditions for regularity of Leray-Hopf solutions to the Navier-Stokes equations by prescribing a condition on only one direction of the velocity field, instead of the full gradient, as in previous work. *S. I. Chernyshenko, P. Constantin, J. C. Robinson, and E. S. Titi* examine the use of Galerkin approximations to the three-dimensional Navier-Stokes equations in view of the fact that there is no rigorous proof of existence and uniqueness in this setting. Thus, they introduce the possibility of rigorous computations of the three-dimensional Navier-Stokes equations to, for example, rule out the possible occurrence of singularity formation in particular examples. They show that if there is a strong solution of the problem, then this can be verified numerically using an algorithm that is guaranteed to terminate in finite time. The work of *K. Ohkitani* considers a class of nonstationary axisymmetric solutions to the Navier-Stokes equations, proving a blow-up theorem for this class of solutions. *D. Cordoba, F. Gancedo, and R. Orive* study the analytical structure of two-dimensional porous media flow with Darcy's law dependence. They obtain local existence and uniqueness theorems and present global existence criteria. Finite-time blow-up results are shown for a class of solutions with infinite energy. *A. Cheskidov, C. R. Doering, and N. P. Petrov* obtain rigorous estimates on the rate of energy dissipation for a variety of body-forced three-dimensional Navier-Stokes equations, obtaining insight on the dependence of the driving force. Particularly interesting is the fact that the upper limit for the dissipation rate may diverge at high Reynolds numbers, which is consistent with recent studies of "fractal forced" turbulence. The manuscript of *G. P. Galdi* describes how the weak formulation of the steady-state Navier-Stokes equations in the exterior of a three-dimensional domain can be formulated as an equivalent problem in an appropriate Banach space. Using this formulation, he is able to prove several bifurcation and stability results pertaining to the original problem. The paper of *B. Birnir, S. Hou, and N. Wellander* considers an incompressible model for airflow through the compression system in turbomachines. Using a multiscale homogenization formalism, they analyze and numerically treat the first several orders of approximation to the full system.

In the second category, **Vanishing Viscosity Limits and the Relation Between Navier-Stokes and Euler Systems**, *C. Y. Jung and R. Temam* obtain asymptotic estimates on solutions of a class of singularly perturbed one-dimensional convection-diffusion models with a turning point in the domain interior. These problems are analogous to zero-viscosity limits of the Navier-Stokes equations with turning point behavior, which require matching two asymptotic expansions across the turning point. *C. Marchioro* studies the zero-viscosity limit in the case with sharply concentrated vorticity fronts, treating in detail the case of a smoke ring with large ring radius. *W. Cheng and X. Wang* prove that numerical solutions to the Navier-Stokes equations converge to solutions of the Euler equations as the viscosity and mesh size go to zero, as long as small scales (of order ν/U) in the direction tangential to the boundary are not resolved in the scheme.

The third category, **Mathematical Issues Related to Discrete Vortex and Dynamical Systems Representations of Fluid Flows**, begins with the paper of *H. Aref* which reviews and highlights some of the fascinating and deep mathematical topics associated with point vortex systems. These topics include integrability and nonintegrability of Hamiltonian systems, the classification of relative equilibria and roots of certain polynomial equations, and topics in projective geometry. *D. Blackmore, L. Ting, and O. Knio* consider the perturbed three-vortex problem, in particular, issues related to the persistence of periodic and quasiperiodic orbits, using Kolmogorov-Arnol'd-Moser and Poincaré-Birkhoff type arguments. The work of *A. V. Borisov, I. S. Mamaev. and S. M. Radodamov* considers the mutual interaction of a system of point vortices with a two-dimensional cylinder. When the cylinder has elliptical shape, a single point vortex is shown to undergo chaotic motion in this rich class of Hamiltonian systems. The article by *F. Lekian, S. C. Shadden, and J. E. Marsden* considers the computation of finite-time Lyapunov exponent fields in time-dependent flows as a way of identifying Lagrangian coherent structures in these flows. They generalize their previous work from two dimensions to n dimensions. *K. Julien and E. Knobloch* consider the problem of deriving reduced order models in constrained flows such as rapidly rotating convection, convection in a strong magnetic field, and magnetorotational instability in accretion disks.

The fourth category, **Statistical Fluid Dynamics and Turbulence Models**, contains the contribution of *A. Majda and X. Wang* in which the authors consider the emergence of large scale structures in two-dimensional flows as a result of small scale random bombardments that occur at discrete time intervals, a generalization of their earlier work where the forcing was modeled as a continuous time random process. The article by *M. Oliver and O. Buhler* considers the evolution of passive scalars in shear flow, which is modeled as a system of lattice differential equations in wave number space. *A. Cheskidov, S. Friedlander, and N. Pavlovic* consider some properties of an infinite-dimensional system of ordinary differential equations that have certain similarities to the inviscid equations of fluid dynamics. They are able to confirm Onsager's conjecture for this model system. The article by *J. S. Linshiz and E. S. Titi* presents an analytical study of a subgrid scale turbulence model of the three-dimensional Magnetohydrodynamics equations with α regularization.

In the fifth category, **Geophysical Fluid Dynamics Models**, the paper of *M. Jamaloodeen and P. K. Newton* formulates the two-layer quasigeostrophic potential vorticity model as both an infinite-dimensional Hamiltonian system and a finite-dimensional system using a discrete point vortex representation of the flow. *C.-H. Hsia, T. Ma, and S. Wang* study bifurcations in bouyancy-driven flows describing thermal convection near the first critical Rayleigh number, both for Prandtl number greater than one and Hopf bifurcation to periodic solutions for Prandtl number smaller than one. The paper of *C. C. Lim* formulates a new variational principle based on extremizing the fixed frame kinetic energy under constant relative enstrophy for flow on a rotating sphere.

Taken as a whole, these articles on **Mathematical Fluid Dynamics** represent just a sample—a current snapshot—of the exciting range of research in this field of applied mathematics. We hope that this special issue of *Journal of Mathematical Physics* will serve as a roadmap to guide future investigations.