Special Issue

Integrability, Topological Solitons and Beyond

Guest Editors

Athanassios S. Fokas

and

Nicholas S. Manton

University of Cambridge, Cambridge, UK

Editor

Roger G. Newton

Indiana University, Bloomington, IN

Journal of Mathematical Physics

Volume 44, Issue 8, August 2003

AMERICAN INSTITUTE OF PHYSICS

INTRODUCTION

This issue brings together recent work on a range of exactly and almost exactly solvable models. The common theme in many of these models is the existence of solitons. Solitons have remarkable stability properties and dominate the long-time asymptotics of solutions of several types of integrable PDE. Solitons also arise in a number of field theories having application in elementary particle physics, where their stability is of a topological kind. Here the equation for static soliton and multi-soliton fields is sometimes integrable, but the dynamics of solitons is usually not. The paradigm integrable example is the self-dual Yang–Mills equation, defined in four space dimensions, with its multi-instanton solutions. Dimensional reduction of the self-dual Yang–Mills equation leads to monopoles and many other solitons.

The first section discusses classical integrable equations with emphasis on the analysis of such equations. Ablowitz, Chakravarty, and Halburd elucidate the remarkable fact that many integrable equations (such as the standard soliton PDEs, the classical Painlevé ODEs, and integrable generalizations of the Darboux–Halphen system and Chazy equations) are exact reductions of the self-dual Yang–Mills equation. Grinevich and Novikov present some new developments in the algebro-geometrical theory of integrable PDEs, which make it possible to solve the long-standing open problem of constructing *real* finite-gap solutions of the sine-Gordon equation as well as computing the associated topological charge. The articles of Boutet de Monvel and Kotlyarov, of Sabatier, and of Fokas discuss recent progress made in the solution of initial-boundary value problems for evolution PDEs both in one and in two spatial dimensions. It is noted that the analysis of boundary value problems (which is much more complicated than the analysis of the standard initial value problems) reveals that solitons (and dromions) continue to play a fundamental role, in the sense that they still dominate long-time asymptotics. Kac and van de Leur extend the fermionic formulation of the KP equation to the *n*-component KP hierarchy; this beautiful formulation, which includes several well known systems (such as the DS system) can be used to construct explicit solutions via vertex operators. Alonso and Mañas use the twistor formalism to construct explicit symmetries as well as to characterize solutions of the dispersionless KP hierarchy. Boiti, Pempinelli, Pogrebkov, and Prinari present the so-called resolvent formalism, which in addition to being very useful for analyzing decaying solutions for integrable PDEs in two spatial dimensions, has the distinctive feature that it can be used for the analysis of non-decaying solutions (such as solutions which are perturbations of a line soliton).

The second section also discusses classical integrable equations but the emphasis is now on both analysis and geometry. Rogers and Schief review the role of geometric constraints in identifying as well as analyzing integrable systems in a wide variety of physical situations in hydrodynamics, magnetohydrodynamics, and elasticity. Grundland and Zakrzewski use known nonholomorphic, harmonic maps to study the Weierstrass problem of immersed surfaces. Dancer and Wang show how to apply Painlevé analysis to a special class of symmetric Einstein spaces, relevant in string theory. Neugebauer and Meinel use powerful analytical tools to solve boundary value problems describing rotating black holes as well as disks of dust (''galaxies''). Dunajski and Mason, by using the twistor formalism, describe interesting geometrical structures in the hyper-Kähler equations. Discrete integrable systems and associated discrete geometrical notions have blossomed in the last decade; the article of Agafonov and Bobenko presents some striking examples of this remarkable development.

The section on Topological Solitons has articles on various types of static and dynamical solitons, living in various geometric settings. In many examples, there are exact static solutions whose moduli (parameter) space has a Riemannian structure, but the metric is often difficult to compute. Soliton dynamics can be approximated by an adiabatic motion through moduli space, usually along geodesics. Haskins and Speight, Baptista and Manton, and also Battye, Gibbons, Rychenkova, and Sutcliffe consider tractable examples of this geodesic motion, and also the accuracy of the approximation. Tong, and also Murray and Singer discuss the geometry of moduli space itself for, respectively, a novel type of vortex, and a large class of monopoles. Battye, Houghton, and Sutcliffe discuss a new class of symmetric Skyrmions, and construct solutions numerically. Ward, and also Ioannidou and Vlachos consider aspects of the mechanism by which otherwise unstable solitons are stabilized by a steady internal rotation, giving them a Noether charge Q.

The final section includes articles on a range of physical effects associated with integrable models and soliton solutions. Aspects of the theory of quantum integrable particle systems and the role of the Yang–Baxter equation are discussed by Hikami and Wadati. Mezincescu and Mezincescu calculate the quantum states of a particle bound to a monopole in a supersymmetric context. Goldhaber discusses more generally the exotic charge assignments and spins of quantized solitons and of bound particle-soliton states. Klinkhamer and Rupp consider a class of unstable static solutions in the electroweak gauge field theory, called sphalerons rather than solitons, and the exotic physical effects of fermions coupled to these. Davies, Hollowood, and Khoze study more general quantum field theory effects of monopoles in supersymmetric gauge theories. Techniques developed in the theory of integrable systems have had a major impact in the theory of orthogonal polynomials and random matrices, as discussed by Baik, Deift, and Strahov.

A. S. Fokas and N. S. Manton
University of Cambridge, Cambridge, UK