Physics Challenge for Teachers and Students

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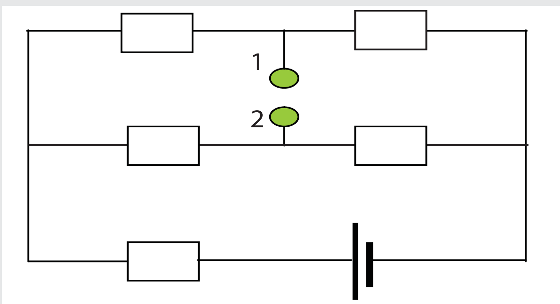
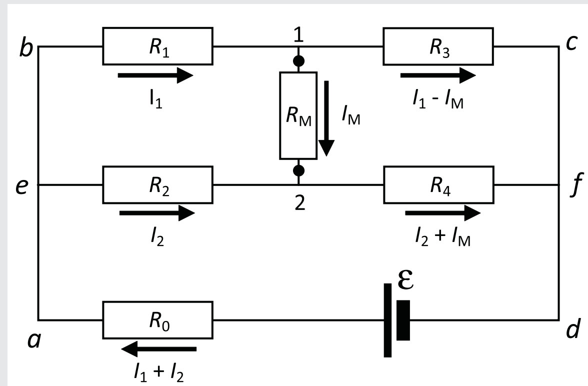
Solution to April 2009 Challenge

Fit for an I-V League

A circuit consists of an emf source and five resistors with unknown resistances. When an ideal ammeter is connected to points 1 and 2, its reading is *I*. When instead a resistor *R* is connected to the same two points, the current through that resistor is *i*. What would be the reading *V* of an ideal voltmeter if it is connected to points 1 and 2 instead of the resistor?





$R_M = 0$ for an ideal ammeter, and $I_M = 0$ for an ideal voltmeter placed between the two terminals. By applying Ohm's law (or Kirchhoff's voltage law) around the loop *abcda* and the loop *aefda*, we obtain the following equations,

$$\begin{cases} \varepsilon = (I_1 + I_2)R_0 + I_1R_1 + (I_1 - I_M)R_3 = (R_0 + R_1 + R_3)I_1 + I_2R_0 - I_MR_3 \\ \varepsilon = (I_1 + I_2)R_0 + I_2R_2 + (I_2 + I_M)R_4 = (R_0 + R_2 + R_4)I_1 + I_1R_0 + I_MR_4 \end{cases} . \quad (1)$$

This *Challenge* attracted a very large number of contributions, most of them correct. Some of the solvers worked through Kirchhoff's rules, whereas others opted to use Thévenin's theorem (which is rarely covered in introductory physics courses). What follows are two solutions that used these two different methods.

Solution 1:

Let us call the resistances R_0, R_1, R_2, R_3, and R_4 and the emf source ε, as shown in the diagram. Imagine also there is a generic resistance R_M between terminals 1 and 2. The assumed electric current flowing through each resistor is shown by an arrow next to the resistor. Later we will set

The writing will be simplified if we introduce an abbreviation $R_{ijk\ldots} \equiv R_i + R_j + R_k \ldots$; with this notation, the equations above become

$$\begin{cases} R_{013}I_1 + R_0I_2 = \varepsilon + R_3I_M \\ R_0I_1 + R_{024}I_2 = \varepsilon - R_4I_M \end{cases} . \quad (2)$$

From here we can solve for I_1 and I_2 in terms of ε, I_M and all the *R*s by using the usual elimination process. For example, we can multiply the first equation with R_{024} and the second with R_0, and then subtract the second equation from the first; we obtain

$$I_1 = \varepsilon \frac{R_{24}}{R_0R_{1234} + R_{13}R_{24}} + I_M \frac{R_0R_{34} + R_3R_{24}}{R_0R_{1234} + R_{13}R_{24}} . \quad (3)$$

In the same way, if we multiply the first equation with R_0 and the second with R_{013}, and then subtract the second equation from the first, we can solve for I_2 to get

$$I_2 = \varepsilon \frac{R_{13}}{R_0 R_{1234} + R_{13} R_{24}} - I_M \frac{R_0 R_{34} + R_4 R_{13}}{R_0 R_{1234} + R_{13} R_{24}}. \quad (4)$$

The potential difference between terminal 1 and 2 is $V_{12} = V_{e2} - V_{b1} = I_2 R_2 - I_1 R_1$. Substituting the currents I_1 and I_2 above, we can write V_{12} in terms or ε, I_M and all the R's,

$$V_{12} = \varepsilon \left(\frac{R_2 R_{13} - R_1 R_{24}}{R_0 R_{1234} + R_{13} R_{24}} \right) - I_M \left(\frac{R_0 R_{12} R_{34} + R_1 R_3 R_{24} + R_2 R_4 R_{13}}{R_0 R_{1234} + R_{13} R_{24}} \right). \quad (5)$$

Overlooking the details, this potential difference can be written as

$$V_{12} = \varepsilon A - I_M B, \quad (6)$$

where the coefficients A and B represent the factors inside the parentheses which depend only on the resistances in the circuit.

When an ideal ammeter ($R_M = 0$) is placed between terminals 1 and 2, the potential difference is $V_{12} = 0$ and the current reading is $I_M = I$. With this information, Eq. (6) yields

$$\varepsilon A = IB. \quad (7)$$

Next, when a resistance R is placed between terminals 1 and 2, $I_M = i$ and $V_{12} = i_R$; Eq. (6) now yields

$$iR = \varepsilon A - iB = IB - iB. \quad (8)$$

Here we have used Eq. (7) to replace εA with IB. Solving for B, we get

$$B = \frac{iR}{I - i}. \quad (9)$$

Finally, when an ideal voltmeter is placed between terminals 1 and 2, $I_M = 0$, and so the reading of the voltmeter is $V_{12} = \varepsilon A - 0 = IB$, or

$$V_{12} = \frac{I\,i}{I - i} R. \quad (10)$$

(Submitted by Erwin Sucipto, Bethel College, Mishawaka, IN)

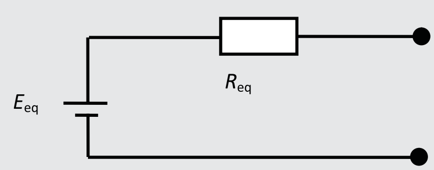
Solution 2.

The "linear superposition principle" allows us to state that if we choose two points (1 and 2) of an electric circuit composed by ***linear elements*** (sources, resistors, capacitors, and inductors), the behavior from the viewpoint of the load connected to those points is equivalent to a simple circuit composed by an ideal voltage source in series with an impedance. (It is known as the Thévenin's equivalent circuit. There is also another equivalent circuit, composed by an ideal current source in parallel with an impedance, the Norton's equivalent circuit.)

Therefore, for our dc circuit we will have:

$$I = \frac{E_{eq}}{R_{eq}},$$

$$i = \frac{E_{eq}}{R + R_{eq}},$$



and

$$E_{eq} = \frac{RIi}{(I - i)} \quad \text{and} \quad R_{eq} = \frac{Ri}{(I - i)}.$$

Therefore, the reading of an ideal voltmeter connected to points 1 and 2 will be obviously:

$$V = \frac{RIi}{(I - i)}.$$

This problem suggests an interesting procedure for obtaining the Thévenin's (or Norton's) equivalent in electric circuits. It is enough to experimentally measure I (with an ideal ammeter connected to points 1 and 2) and i for a known resistor R (or impedance Z in ac circuits) between these points.

(Submitted by José Ignacio Íñiguez de la Torre, Universidad de Salamanca, Salamanca, Spain)

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Many thanks to all contributors and we hope to hear from you in the future.

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