Physics Challenge for Teachers and Students

***Boris Korsunsky*, Column Editor**
Weston High School, Weston, MA 02493
korsunbo@post.harvard.edu

Solution to December 2009 Challenge

Happy New Year

Imagine that the mass of the Sun instantly doubles. How long would the Earth's year be?

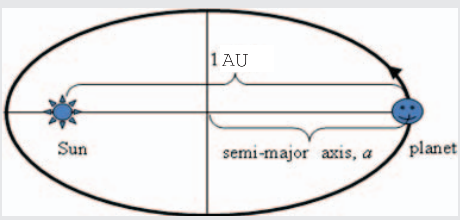
(Contributed by Carl E. Mungan, U.S. Naval Academy, Annapolis, MD)

Solution:

I assume that to a good approximation, the Earth's orbit is circular with a radius a_1 = 1 AU, so that the constant kinetic energy and gravitational energy are related easily. I assume that the Sun's mass M_s >> Earth's mass m. Therefore the orbital period P in years of the Earth around the Sun is given by Kepler's third law $P^2 = a^3/M$, where a is in astronomical units and M is the Sun's mass in solar units (currently = 1.0). So the problem of finding the Earth's new period becomes one of finding the new semi-major axis after the Sun's mass has doubled.

1). A quick lower limit to the new period

When the Sun's mass doubles, the stronger centripetal force on the Earth will cause it to curve inwards into a new, elliptical orbit, with the Sun at the far focal point, a smaller semi-major axis and period, and Earth's position now marking the new orbit's aphelion. Sketching the new orbit, it would look something like this:



The aphelion Earth-Sun distance (= 1 AU in this case) must always be less than $2a$, the major axis of the ellipse. So we can be sure that the new semi-major axis a > ½ AU, no matter what the mass increase of the Sun (as long as the new orbit doesn't cause a collision with the Sun, of course).

And since the orbit just got smaller, we also know that a < 1 AU. Applying Kepler's third law to this lower limit on a, with the new central mass M = 2 solar units, means that the new period

$$P > \sqrt{\frac{\left(\frac{1}{2}\right)^2}{2}}, \text{ i.e.}$$

¼ < P < 1 yr.

(I tried to come up with a more interesting upper limit on P, using the fact that Earth's angular momentum is unchanged, but could not.)

2. The exact solution

To find the new period, Kepler's third law tells us we just need to find the new value of a; we don't have to be concerned with the orbit's shape (eccentricity). I recall that the conserved mechanical energy E of an elliptical orbit (= kinetic + potential energy, i.e. $E = ½\, mv^2 - GMm/r$) is also independent of eccentricity and depends only on a, so if we find the new orbital energy after the Sun's mass doubles, we should be able to derive the corresponding new value of a.

For a circular orbit it is easy to show from centripetal force arguments that the kinetic energy K is given by $K = ½\, mv^2 = ½GM_sm/a_1$, while the gravitational potential energy $U = -GM_sm/a_1$.

So the mechanical energy of the orbit is

$$E = K+U = GM_sm/2a_1 - GM_sm/a_1 = -GM_sm/2a_u,$$

where for the Earth of mass m in kg, a_1 = 1 AU in meters and M_s= 1 solar mass in kg. Now suddenly doubling the Sun's mass will double the Earth-Sun's (negative) gravitational potential energy, without changing its kinetic energy, so the new mechanical energy of the Earth-Sun system is now:

$$E_{new} = ½GM_sm/a_1 - G(2M_s)m/a_u = -3/2GM_sm/a_1.$$

Equating this with the general formula for mechanical energy of a planet revolving around a star of mass $2M_s$ with a new semi-major axis a_2,

$$E_2 = -G(2M_s)m/2a^2 ,$$

we see that we must have $a_2 = 2/3\ a_1$.

So the Earth's new orbital semi-major axis is now 2/3 AU. As a check, this falls in the $\frac{1}{2} < a < 1.0$ AU range we found earlier.

From this, using Kepler's third law $P^2 = a^3/M$, with M = 2 solar masses, the new period is now

$$P = \sqrt{\frac{(\frac{2}{3})^3}{2}} = \frac{2}{3\sqrt{3}} = 0.385 \text{ yr},$$

again within the range $\frac{1}{4} < P < 1.0$ yr found earlier.

Using the standard formulae for orbits, it can be shown that the new orbital eccentricity $e = \frac{1}{2}$. This has no effect on the period but might be a number of interest to the hapless denizens of Earth as they are pulled in toward the Sun in their new orbit, since the perihelion distance = $a_2(1-e)$ = 1/3 AU — much too close for comfort! (At least the Earth will be moving three times faster at this point compared to its previous aphelion velocity.)

(Contributed by Philip Blanco, Grossmont College, El Cajon, CA)

We are also pleased to recognize the following contributors:

Bruce Barnett (Johns Hopkins University, Baltimore, MD)

Daniel Cartin (Naval Academy Preparatory School, Middetown, RI)

R. C. Dhandhania (KalraShukla, Mumbai, India)

Don Easton (Lacombe, Alberta, Canada)

Hasan Fakhruddin (The Indiana Academy for Science, Mathematics, and Humanities, Ball State University, Muncie, IN)

Fernando Ferreira (Universidade da Beira Interior, Covilhã, Portugal)

Fredrick P. Gram (Cuyahoga Community College, Cleveland, OH)

David Grinstein (Waltham, MA)

Art Hovey (Milford, CT)

J. Iñiguez (Universidad de Salamanca, Salamanca, Spain)

David Jones (Florida International University, Miami, FL)

Mark Lenfestey (Homestead High School, Fort Wayne, IN)

Stephen McAndrew (Trinity Grammar School, Summer Hill, NSW, Australia)

Jeff Melmed (Eastern Maine Community College, Bangor, ME)

Matthew W. Milligan (Farragut High School, Knoxville, TN)

Daniel Mixson (Naval Academy Preparatory School, Newport, RI)

Bill Nettles (Union University, Jackson, TN)

Gregory Ruffa (University of Minnesota, Minneapolis, MN)

Daniel Schumayer (University of Otago, Dunedin, New Zealand)

Jason L. Smith (Richland Community College, Decatur, IL)

Cássio dos Santos Sousa, student (Colégio Objetivo, São Paulo, Brazil)

Many thanks to all contributors and we hope to hear from you in the future!

Please send correspondence to:

Boris Korsunsky

korsunbo@post.harvard.edu