# Physics Challenge for Teachers and Students

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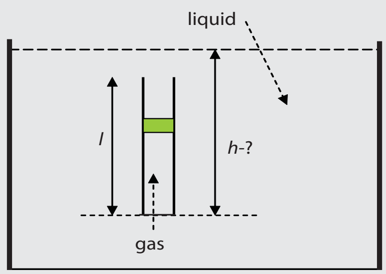
## Solution to January 2009 Challenge

### Phd in the Water

*Challenge:* A test tube of length $l$ is filled with gas at pressure $P$ and is then sealed with a light, movable piston. The test tube is then immersed vertically into a liquid of density $d$ so that the bottom end is a distance $h$ from the surface of the liquid. What is the minimum value of $h$ that allows the piston to stay in equilibrium inside the test tube as shown? The atmospheric pressure is $P_a$. The temperature of the gas remains constant.




*Solution:* I assume that the tube is being held immersed in the fluid at any desired depth by a clamp that is not shown. Also I assume the gas is ideal, so that the product of its pressure $p$ and volume $Ax$ (where $x$ is the height of the piston above the bottom of the test tube) is constant, equal to its initial value of $PAl$ when it was sealed, where $A$ is the cross-sectional area of the test tube. Therefore, the pressure of the gas when the piston is at height $x$ is

$$p = Pl / x \,. \tag{1}$$

There are three possible cases to consider. The first possibility is that the initial gas pressure is less than or equal to atmospheric pressure, $P \leq P_a$. In that case, the atmosphere will push the piston into the tube until $x = Pl / P_a$ when equilibrium is reached, according to Eq. (1). That is, no fluid immersion is required at all, so that $h_{min} = 0$. This is the trivial case. The more interesting possibilities occur for $P > P_a$, as we shall next consider. The second possibility is another special limiting case that arises when the piston is at the top end of the tube, so that $x = l$. In that case, the pressure of the gas inside the tube equals its initial value of $P$. Force balance on the piston then requires that the gas pressure on its bottom face equals the fluid pressure on its top face,

$$P = P_a + (h - l)gd. \tag{2}$$

Call the value of $h$ that satisfies this equation $h_1$,

$$h_1 = \frac{P - P_a}{gd} + l. \tag{3}$$

Since $P > P_a$, $h_1$ must be larger than $l$ and so it is always possible to equilibrate the tube by immersing its bottom end to this depth. This value of $h_1$ therefore will be the minimum value of $h$ under appropriate conditions.

However, depending on the values of the various parameters in Eq. (3), it is not hard to show that we may be able to get a smaller value of $h$ by raising the tube and allowing the fluid to push the piston down to a height $x$ less than $l$. The pressure in the tube is then given by Eq. (1) and must balance the hydrostatic pressure at the depth $h - x$ of the piston below the liquid surface, so that we must replace Eq. (2) with

$$Pl / x = P_a + (h - x)gd, \tag{4}$$

which rearranges into

$$h = \frac{Pl}{xgd} - \frac{P_a}{gd} + x. \tag{5}$$

We minimize this by setting $dh / dx = 0$ to find the optimal value of $x$, namely

$$x_{\text{best}} = \sqrt{\frac{Pl}{gd}}, \tag{6}$$

which is allowable provided that $x_{\text{best}} < l$ (so that the piston remains in the tube at a lower height than for the second possibility above). In turn, this inequality holds if $P < lgd$. Provided that condition holds, then we can substitute Eq. (6) into (5) to obtain the minimal value of $h$,

$$h_2 = 2\sqrt{\frac{Pl}{gd}} - \frac{P_a}{gd}. \tag{7}$$

It remains to verify that $h_2 < h_1$ so that this third case is an improvement on the second case. One elegant way to perform this verification is to first derive the dimensionless relation

$$\frac{h_1 - h_2}{l} = \left(1 - \sqrt{\frac{P}{lgd}}\right)^2 \tag{8}$$

using Eqs. (3) and (7). Then observe that the right-hand side can never be negative to conclude that $h_2$ must be smaller than $h_1$. (In fact, case 3 would therefore always be superior to case 2 if it were not for the fact that $x$ cannot exceed $l$ in value!)

**To summarize the three possibilities:**

1. If $P \le P_a$, then $h_{\min} = 0$.
2. If $P$ is larger than both atmospheric pressure, $P_a$, and the hydrostatic pressure difference across the length of the tube, $lgd$, then $h_{\min} = h_1$.
3. Finally, if $lgd > P > P_a$, then $h_{\min} = h_2$.

As a check, note from Eq. (8) that $h_1 = h_2$ if $P = lgd$, so that the second and third cases then reduce to the same solution.

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