Physics Challenge for Teachers and Students

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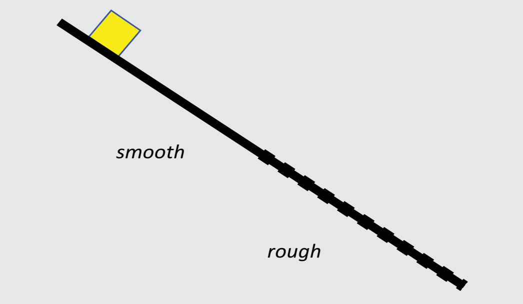
Solution to September 2009 Challenge

Half and Rough

Challenge: A small block slides down a slanted board when released. The upper half of the board is smooth and the lower is rough, so that the acceleration of the block on the smooth half is three times greater than it is on the rough half. The block reaches the bottom of the board in time t_1. The board is then flipped so that the upper half is rough and the lower part is smooth, and the block is released from the top again. This time, the block reaches the bottom of the board in time t_2. In both cases, the board makes the same angle with the horizontal. Find the ratio t_1/t_2.



Column Editor's note: Many readers went "above and beyond" finding the answer and provided the comments that would no doubt be useful and interesting to anyone seeking to extract the most out of this problem. As an exception to the usual format, I will also post some of those "extra" comments following the solution.

Solution: We analyze the general situation calling each half of the total distance L and the acceleration on the two surfaces to be A and NA, where N is a positive constant such that $0 < N \leq 1$ (indicating that the rough surface comes second and has smaller acceleration). Using constant acceleration kinematics for each of the two parts of the motion, we have for the first distance L in time T:

$$v = v_0 + at \quad \rightarrow \quad v = AT. \qquad (1)$$

$$\Delta x = v_0 \Delta t + \frac{1}{2} a \Delta t^2 \quad \rightarrow \quad L = \frac{1}{2}(A)(T)^2 \rightarrow$$
$$T = \sqrt{\frac{2L}{A}}. \qquad (2)$$

Now, for the second part of the motion, we have (with t_B meaning total time to the bottom)

$$\Delta x = v_0 \Delta t + \frac{1}{2}(a)\Delta t^2 \quad \rightarrow \quad L = (AT)(t_B - T)$$
$$+ \frac{1}{2}(NA)(t_B - T)^2. \qquad (3)$$

Substituting Eq. (2) into Eq. (3) and rearranging with some algebra leads to

$$\frac{1}{2} NA\, t_B^2 + (1 - N)\sqrt{2LA}\, t_B + (N-3)L = 0,$$

which has positive physical solution

$$t_B = \frac{(N-1)\sqrt{2LA} + \sqrt{2LA(N-1)^2 - 4(L)(N-3)(\frac{1}{2}NA)}}{NA}$$
$$= \sqrt{\frac{2L}{A}}\left(\frac{N - 1 + \sqrt{N+1}}{N}\right). \qquad (4)$$

By choosing the acceleration for the first trip on the first surface to be A with the second half of the trip having acceleration NA, when the reverse trip is considered, the substitutions required into Eq. (4) to find the time would be to let $N \rightarrow (1/N)$ and then substitute $A \rightarrow (NA)$.

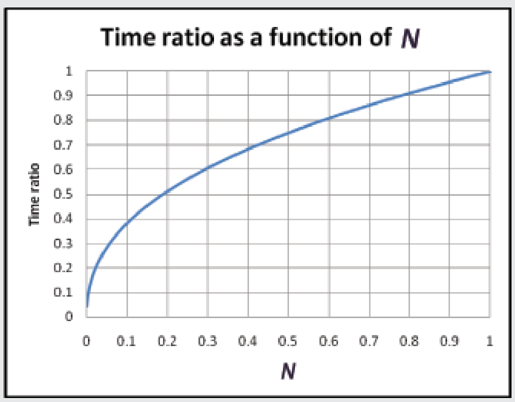
With this information, the ratio of the times for smooth start to rough start becomes

$$\frac{t_1}{t_2} = \frac{\sqrt{\frac{2L}{A}\left(\frac{N-1+\sqrt{N+1}}{N}\right)}}{\sqrt{\frac{2L}{NA}\left(\frac{\frac{1}{N}-1+\sqrt{\frac{1}{N}+1}}{\frac{1}{N}}\right)}}$$

$$= \sqrt{\frac{1}{N}\left(\frac{N-1+\sqrt{N+1}}{N}\right)\left(\frac{\frac{1}{N}}{\frac{1}{N}-1+\sqrt{\frac{1}{N}+1}}\right)}$$

$$= \frac{1}{\sqrt{N}}\left(\frac{N-1+\sqrt{N+1}}{1-N+\sqrt{N^2+N}}\right).$$

The graph shows the ratio of times for this problem as a function of the value of N.



For the problem of interest, we have $N = 1/3$, leading to a time ratio of

$$\frac{t_1}{t_2} = \frac{1}{\sqrt{\frac{1}{3}}}\left(\frac{\frac{1}{3}-1+\sqrt{\frac{1}{3}+1}}{1-\frac{1}{3}+\sqrt{\frac{1}{9}+\frac{1}{3}}}\right)$$

$$= \sqrt{3}\left(\frac{-\frac{2}{3}+\sqrt{\frac{4}{3}}}{\frac{2}{3}+\sqrt{\frac{4}{9}}}\right) = \sqrt{3}\,\frac{2(-1+\sqrt{3})}{4} = \frac{3-\sqrt{3}}{2}\quad.$$

(Contributed by Michael C. Faleski, Delta College, Midland, MI)

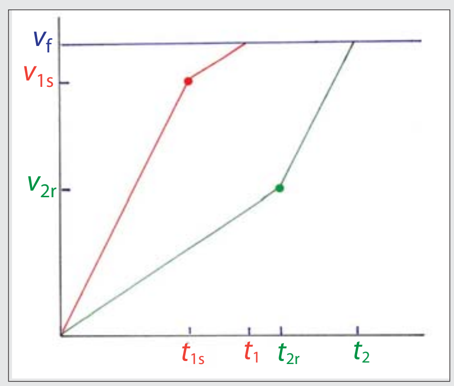
Comments, addition, alternatives:

1. One can draw free-body diagrams of the block on each half of the board to straightforwardly show that the coefficient of kinetic friction on the rough section must be $\mu_k = \frac{2}{3}\tan\theta.$ Recall that coefficients of static friction are larger than the corresponding coefficients of kinetic friction. We must assume, however, that the coefficient of static friction is smaller than $\mu_{s,\,max} = \tan\theta$ as otherwise the block would not begin to slide when placed at rest on the rough end of the board. But other than that, we do not need to know the value of the coefficients of friction in order to solve the problem.

2. Not only did the solution not require knowledge of the values of μ, it does not in fact require knowledge of a_S and a_R separately but only of their ratio $a_S/a_R = 3$. For example, there could be friction on both halves of the board, but with one half rougher than the other such that the ratio of accelerations remains 3. The value of t_1/t_2 in Eq. (8) would then remain unchanged.

(Contributed by Carl E. Mungan, U.S. Naval Academy, Annapolis, MD)

The kinematical analysis can also be conducted entirely in terms of velocities. On a velocity-versus-time graph, the paths representing the two slides form the sides of a parallelogram. We will be interested in the times at which the velocity for each slide reaches its final value.



If we use the acceleration a_s which the block experiences on the "smooth" section as a reference, and the length of each section is L, the block starting from rest undergoes increases in velocity on the smooth or rough sections given by

$$(\Delta v)_s^2 = 2a_s L \quad , \quad (\Delta v)_r^2 = 2(\tfrac{1}{3}a_s)L = \tfrac{2}{3}a_s L.$$

For the slide with the smooth section encountered first, the final velocity is given by

$$v_f^2 = (\Delta v)_s^2 + 2(\tfrac{1}{3}a_s)L,$$

while for the slide with the rough section encountered first, the final velocity is found from $v_f^2 = (\Delta v)_r^2 + 2a_s L$. Hence, for both slides, the block, which starts from rest, reaches the bottom with final velocity

$$v_f = \sqrt{\tfrac{8}{3} a_s L}.$$

(This also follows from work-energy considerations.)

(Contributed by Gregory Ruffa, University of Minnesota, Minneapolis, MN)

Assuming that the accelerations are non-zero, but their ratio, $\Lambda = a_1/a_2$, is extremely large, the acceleration on the top half of the board is much larger than on the rough side, i.e. $\Lambda >> 1$.

$$\frac{t_1}{t_2} = \frac{\sqrt{\frac{1}{\Lambda}}(1-\Lambda)+\sqrt{1+\Lambda}}{1-\frac{1}{\Lambda}+\frac{1}{\Lambda}\sqrt{1+\Lambda}} \approx \frac{\frac{1}{\sqrt{\Lambda}}-\Lambda\sqrt{\frac{1}{\Lambda}}+\sqrt{1+\Lambda}}{1}$$
$$= \frac{1}{\sqrt{\Lambda}}+\sqrt{1+\Lambda}-\sqrt{\Lambda} \cong \frac{3}{2\sqrt{\Lambda}} << 1 \qquad (7)$$

(Contributed by Daniel Schumayer, University of Otago, Dunedin, New Zealand)

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Many thanks to all contributors and we hope to hear from you in the future!

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